Severi-Bouligand tangents, Frenet frames and Riesz spaces

SEVERI-BOULIGAND TANGENTS, FRENET FRAMES AND RIESZ SPACES

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ABSTRACT. A compact set $X \subseteq \mathbb{R}^2$ has an outgoing Severi-Bouligand tangent unit vector u at some point $x \in X$ iff some principal quotient of the Riesz space $\mathcal{R}(X)$ of piecewise linear functions on X is not archimedean. To generalize this preliminary result, we extend the classical definition of Frenet k-frame to any sequence $\{x_i\}$ of points in $\mathbb{R}^n$ converging to a point x, in such a way that when the $\{x_i\}$ arise as sample points of a smooth curve γ, the Frenet k-frames of $\{x_i\}$ and of γ at x coincide. Our method of computation of Frenet frames via sample sequences of γ does not require the knowledge of any higher-order derivative of γ. Given a compact set $X \subseteq \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, a Frenet k-frame u is said to be a *tangent* of X at x if X contains a sequence $\{x_i\}$ converging to x, whose Frenet k-frame is u. We prove that X has an outgoing k-dimensional tangent of X iff some principal quotient of $\mathcal{R}(X)$ is not archimedean. If, in addition, X is convex, then X has no outgoing tangents iff it is a polyhedron.

1. Introduction

In [10, §53, p.59 and p.392] and [11, §1, p.99], Severi defined (outgoing) tangents of arbitrary subsets of the euclidean space $\mathbb{R}^n$. Subsequently and independently, Bouligand defined the same notion [2, p.32], which today is widely known as "Bouligand tangent". Throughout we will adopt the following equivalent definition, where $||\cdot||$ denotes euclidean norm and $\mathrm{conv}(Y)$ is the convex hull of $Y \subseteq \mathbb{R}^n$:

Definition 1.1. [8, pp.14 and 133] Let $\emptyset \neq X \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. A unit vector $u \in \mathbb{R}^n$ is a *Severi-Bouligand tangent of X at x* if X contains a sequence $\{x_i\}$ such that $x_i \neq x$ for all i, $\lim_{i\to\infty} x_i = x$, and $\lim_{i\to\infty} (x_i - x)/||x_i - x|| = u$. If for some $\mu > 0$, $\mathrm{conv}(x, x+\mu u) \cap X = \{x\}$, we say that u is *outgoing*.

For an equivalent algebraic handling of tangents, in Section 4 we introduce the Riesz space (=vector lattice) $\mathcal{R}(X)$ of piecewise linear functions on any nonempty compact set $X \subseteq \mathbb{R}^n$. When $n = 2$, the geometric properties of X are immediately linked to the algebraic properties of $\mathcal{R}(X)$ by the following elementary result (Lemma 4.3): *If $\mathcal{R}(X)$ has a non-archimedean principal quotient then X has an outgoing Severi-Bouligand tangent.*

In Theorem 5.1 we will extend this result, as well as its converse, to all n. To this purpose, in Section 2 we introduce the notion of a Frenet k-frame *of a sequence* $\{x_i\}$ of points in $\mathbb{R}^n$, as the natural generalization of the classical Frenet (Jordan) k-frame [5, 4] of a curve γ. Specifically, if the x_i arise as sample points of a smooth curve γ accumulating at some point x of γ, then the Frenet k-frame of $\{x_i\}$ coincides with the Frenet k-frame of γ at x. This is Theorem 2.2. The proof *yields a method to calculate the Frenet k-frame of a C^{k+1} curve γ at a point x without knowing the derivatives of any parametrization of γ*: one just takes a sampling sequence $\{x_i\}$ of points of γ converging to x, and then makes the linear algebra calculations as in the proof of the theorem. To show the wide applicability of our method, Example 2.5 provides a curve γ having no Frenet

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k-frame at a point x, but such that the Frenet k-frame of each sequence of points of γ converging to x exists and is independent of the parametrization of γ.

In Section 3 we deal with the relationship between the Frenet k-frame $u = (u_1, \ldots, u_k)$ of a sequence $\{x_i\}$ in $\mathbb{R}^n$ converging to x, and any simplex $T \subseteq \mathbb{R}^n$ containing $\{x_i\}$. Theorem 3.3 shows that T automatically contains the simplex $\mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k)$, for some $\lambda_1, \ldots, \lambda_k > 0$. This elementary result will find repeated use in the rest of the paper.

As a k-dimensional generalization of the classical Severi-Bouligand tangents, we then say that a Frenet k-frame u is *tangent* at x to a compact set $X \subseteq \mathbb{R}^n$ if X contains a sequence $\{x_i\}$ converging to x, whose Frenet k-frame is u. Then Theorem 5.1 provides the desired strengthening of Lemma 4.3, showing that X has no outgoing tangent iff *every principal ideal of* $\mathcal{R}(X)$ *is an intersection of maximal ideals.* This latter property is considered in the literature for various classes of structures: For commutative noetherian rings it is known as "von Neumann regularity"; frames having this property are known as "Yosida frames", [7, 2.1]; Chang MV-algebras with this property are said to be "strongly semisimple", [3]. As a corollary of Stone representation ([6, 4.4]), every boolean algebra is strongly semisimple.

Since $\{+, -, \wedge, \vee\}$-reducts of Riesz spaces with strong unit are lattice-ordered abelian groups with strong unit, and the latter are categorically equivalent to MV-algebras, [9, 3.9], following [3] we say that a Riesz space R is *strongly semisimple* if every principal ideal of R is an intersection of maximal ideals of R. Equivalently, every principal quotient of R is archimedean. A large class of examples of strongly semisimple Riesz spaces with totally disconnected maximal spectrum is immediately provided by hyperarchimedean Riesz spaces, [1]. At the other extreme, when X is a polyhedron, $\mathcal{R}(X)$ is strongly semisimple, (see Proposition 6.2).

Using Theorem 5.1, in Theorem 6.4 we prove that a nonempty compact *convex* subset $X \subseteq \mathbb{R}^n$ has no outgoing tangent iff X has only finitely many extreme points iff X is a polyhedron. This shows the naturalness of Definition 4.1 of "outgoing tangent" as a k-dimensional extension of the classical Severi-Bouligand tangent. Counterexamples of Theorem 6.4 are easily found in case X is not convex (see Example 6.3).

The only prerequisite for this paper is a working knowledge of elementary polyhedral topology (as given, e.g., by the first chapters of [12]), and of the classical Yosida (Kakutani-Gelfand-Stone) correspondence between points of X and maximal ideals of the Riesz space $\mathcal{R}(X)$. See [6] for a comprehensive account.

2. The Frenet frame of a sequence $\{x_i\} \subseteq \mathbb{R}^n$

Given two sequences $\{p_i\}, \{q_i\} \subseteq \mathbb{R}$, by writing $\lim_{i\to\infty} p_i/q_i = r$ we understand that $q_i \neq 0$ for each i, and $\lim_{i\to\infty} p_i/q_i$ exists and equals r.

For any vector $y \in \mathbb{R}^n$ and linear subspace L of $\mathbb{R}^n$, the orthogonal projection of y onto L is denoted

$$\mathsf{proj}_L(y).$$

For our generalization of Severi-Bouligand tangents we first extend Definition 1.1, replacing the unit vector $u \in \mathbb{R}^n$ therein by a k-tuple $\{u_1, \ldots, u_k\}$ of pairwise orthogonal unit vectors in $\mathbb{R}^n$.

Definition 2.1. Given a sequence $\sigma = \{x_i\}$ of points in $\mathbb{R}^n$ converging to x, and a k-tuple $(u_1, \ldots, u_k)$ of pairwise orthogonal unit vectors in $\mathbb{R}^n$, we say:

- u_1 *is the Frenet* 1*-frame of* σ if $u_1 = \lim_{i\to\infty}(x_i - x)/||x_i - x||$;
- $(u_1, \ldots, u_k)$ *is the Frenet k-frame of* σ if $(u_1, \ldots, u_{k-1})$ is the Frenet $(k-1)$-frame of σ, and
$$u_k = \lim_{i\to\infty} \frac{x_i - x - \mathsf{proj}_{\mathbb{R}u_1+\cdots+\mathbb{R}u_{k-1}}(x_i - x)}{||x_i - x - \mathsf{proj}_{\mathbb{R}u_1+\cdots+\mathbb{R}u_{k-1}}(x_i - x)||}.$$

Following [5], for $[a, b] \subseteq \mathbb{R}$ an interval, suppose $\phi\colon [a, b] \to \mathbb{R}^n$ is a C^k function such that for all $a \le t < b$, the k-tuple of vectors $(\phi'(t), \phi''(t), \ldots, \phi^{(k)}(t))$ forms a linearly independent set in $\mathbb{R}^n$. Then the Gram-Schmidt process yields an orthonormal k-tuple $(v_1(t), \ldots, v_k(t))$, called the *Frenet k-frame* of ϕ at $\phi(t)$.

The terminology of Definition 2.1 is justified by the following result:

Theorem 2.2. *Suppose $\phi\colon [a,b] \to \mathbb{R}^n$ is a C^{k+1} function. Let $a \leq t_0 < b$ be such that the vectors $\phi'(t_0), \phi''(t_0), \ldots, \phi^{(k)}(t_0)$ are linearly independent. Then for every sequence $t_1, t_2, \ldots$ in $[t_0, b] \setminus \{t_0\}$ converging to t_0, the Frenet k-frame of $\{\phi(t_i)\}$ exists and is equal to the Frenet k-frame of ϕ at $\phi(t_0)$.*

Proof. We can write

$$\phi(t) = \phi(t_0) + \phi'(t_0)(t - t_0) + \frac{\phi''(t_0)}{2}(t - t_0)^2 + \cdots + \frac{\phi^{(k)}(t_0)}{k!}(t - t_0)^k + R(t)\,, \tag{1}$$

where the remainder $R\colon [a,b] \to \mathbb{R}^n$ satisfies

$$||R(t)|| \leq M(t - t_0)^{k+1} \quad \text{for some } 0 \leq M \in \mathbb{R}. \tag{2}$$

Let $(v_1, \ldots, v_k)$ be the Frenet k-frame of ϕ at $\phi(t_0)$. Then $v_1 = \phi'(t_0)/||\phi'(t_0)||$, and for each $1 < j \leq k$,

$$v_j = \frac{\phi^{(j)}(t_0) - \mathsf{proj}_{\mathbb{R}v_1 + \cdots + \mathbb{R}v_{j-1}}(\phi^{(j)}(t_0))}{||\phi^{(j)}(t_0) - \mathsf{proj}_{\mathbb{R}v_1 + \cdots + \mathbb{R}v_{j-1}}(\phi^{(j)}(t_0))||}.$$

By induction on $1 \leq j \leq k$ we will prove that the Frenet j-frame $(u_1, \ldots, u_j)$ of the sequence $\{\phi(t_i)\}$ (exists and) coincides with the Frenet j-frame $(v_1, \ldots, v_j)$ of ϕ at $\phi(t_0)$.

Basis: Since $||\phi'(t_0)|| \neq 0$, for all suitably large i we have $\phi(t_i) \neq \phi(t_0)$ and

$$\begin{aligned} u_1 &= \lim_{i\to\infty} \frac{\phi(t_i) - \phi(t_0)}{||\phi(t_i) - \phi(t_0)||} \\ &= \lim_{i\to\infty} \frac{(\phi(t_i) - \phi(t_0))/(t_i - t_0)}{||(\phi(t_i) - \phi(t_0))/(t_i - t_0)||} \\ &= \frac{\lim_{i\to\infty}(\phi(t_i) - \phi(t_0))/(t_i - t_0)}{||\lim_{i\to\infty}(\phi(t_i) - \phi(t_0))/(t_i - t_0)||} \\ &= \frac{\phi'(t_0)}{||\phi'(t_0)||} \\ &= v_1. \end{aligned}$$

Induction Step: By induction hypothesis, for each $1 \leq j < k$ the j-tuple $(v_1, \ldots, v_j)$ coincides with the Frenet j-frame $(u_1, \ldots, u_j)$ of the sequence $\{\phi(t_i)\}$. Let the linear subspace S_j of $\mathbb{R}^n$ be defined by

$$S_j = \mathbb{R}u_1 + \cdots + \mathbb{R}u_j = \mathbb{R}v_1 + \cdots + \mathbb{R}v_j = \mathbb{R}\phi'(t_0) + \cdots + \mathbb{R}\phi^{(j)}(t_0).$$

From (2) we have

$$\frac{||R(t) - \mathsf{proj}_{S_j}(R(t))||}{(t - t_0)^{j+1}} \leq M(t - t_0)^{k-j}. \tag{3}$$

For each $l = j+1, \ldots, k$ let us define the vector $\alpha_l \in \mathbb{R}^n$ by

$$\alpha_l = \frac{\phi^{(l)}(t_0) - \mathsf{proj}_{S_j}(\phi^{(l)}(t_0))}{l!}, \tag{4}$$

whence in particular,

$$||\alpha_{j+1}|| = \frac{||\phi^{(j+1)}(t_0) - \mathsf{proj}_{S_j}(\phi^{(j+1)}(t_0))||}{(j+1)!} \neq 0.$$

By (1),

$$\begin{aligned} &\phi(t_i) - \phi(t_0) - \mathsf{proj}_{S_j}(\phi(t_i) - \phi(t_0)) = \\ &\qquad \alpha_{j+1}(t_i - t_0)^{j+1} + \cdots + \alpha_k(t_i - t_0)^k + R(t_i) - \mathsf{proj}_{S_j}(R(t_i)). \end{aligned} \tag{5}$$

From (3)-(5) we get

$$u_{j+1} = \lim_{i\to\infty} \frac{\phi(t_i) - \phi(t_0) - \mathsf{proj}_{S_j}(\phi(t_i) - \phi(t_0))}{||\phi(t_i) - \phi(t_0) - \mathsf{proj}_{S_j}(\phi(t_i) - \phi(t_0))||}$$

$$= \lim_{i\to\infty} \frac{\alpha_{j+1}(t_i - t_0)^{j+1} + \cdots + \alpha_k(t_i - t_0)^k + R(t_i) - \mathsf{proj}_{S_j}(R(t_i))}{||\alpha_{j+1}(t_i - t_0)^{j+1} + \cdots + \alpha_k(t_i - t_0)^k + R(t_i) - \mathsf{proj}_{S_j}(R(t_i))||}$$

$$= \lim_{i\to\infty} \frac{\sum_{l=j+1}^{k} \alpha_l(t_i - t_0)^{l-(j+1)} + (R(t_i) - \mathsf{proj}_{S_j}(R(t_i))) \cdot (t_i - t_0)^{-(j+1)}}{||\sum_{l=j+1}^{k} \alpha_l(t_i - t_0)^{l-(j+1)} + (R(t_i) - \mathsf{proj}_{S_j}(R(t_i))) \cdot (t_i - t_0)^{-(j+1)}||}$$

$$= \frac{\alpha_{j+1}}{||\alpha_{j+1}||} = \frac{\phi^{(j+1)}(t_0) - \mathsf{proj}_{S_j}(\phi^{(j+1)}(t_0))}{||\phi^{(j+1)}(t_0) - \mathsf{proj}_{S_j}(\phi^{(j+1)}(t_0))||} = v_{j+1}.$$

This concludes the proof. □

Remark 2.3. The assumption $\phi \in C^{k+1}$ can be relaxed to $\phi \in C^k$, so long as the kth Taylor remainder $R(t)$ satisfies (2).

Remark 2.4. Theorem 2.2 yields a method to calculate the Frenet k-frame of a C^{k+1} curve, not involving higher-order derivatives, but taking instead a sampling sequence $\{x_i\}$ of points on the curve, and then making the elementary linear algebra calculations in the proof above.

The wide applicability of this method is shown by the following example:

Example 2.5. Let $\phi\colon [0,1] \to \mathbb{R}^2$ be defined by $\phi(x) = (x, x^3)$. Then $\phi'(0) = (1,0)$ and $\phi''(0) = (0,0)$. The Frenet 1-frame of ϕ at $(0,0)$ is the vector $(1,0)$, but ϕ has no Frenet 2-frame at $(0,0)$. And yet, letting $\mathbb{R}(1,0)$ denote the linear subspace of $\mathbb{R}^2$ given by the x-axis, every sequence $\{t_i\} \in [0,1] \setminus \{0\}$ converging to 0 satisfies

$$\lim_{i\to\infty} \frac{\phi(t_i) - \phi(0) - \mathsf{proj}_{\mathbb{R}(1,0)}(\phi(t_i) - \phi(0))}{||\phi(t_i) - \phi(0) - \mathsf{proj}_{\mathbb{R}(1,0)}(\phi(t_i) - \phi(0))||} = \lim_{i\to\infty} \frac{(0, t_i^3)}{||(0, t_i^3)||} = (0,1).$$

We have shown: *There exist a curve γ having no Frenet k-frame at a point x, but the Frenet k-frame of every sequence of points of γ converging to x exists and is independent of the parametrization of γ.*

Example 2.6. While under the hypotheses of Theorem 2.2 the Frenet k-frames of any two sampling sequences of a curve γ at a point $x \in \gamma$ are equal, the map $\psi(x) = (x, x^2 \sin(1/x))\colon [0,1] \to \mathbb{R}^2$ (with the proviso that $\psi(0) = (0,0)$), yields an example of a curve γ that is not C^2 and has two sequences $\{x_i\}$ and $\{y_i\}$ of points of γ both converging to the same point $(0,0)$ of γ, but having different Frenet 2-frames.

3. Simplexes and Frenet frames

Fix $n = 1, 2, \ldots$. For any subset E of the euclidean space $\mathbb{R}^n$, the *convex hull* $\mathrm{conv}(E)$ is the set of all *convex combinations* of elements of E. We say that E is *convex* if $E = \mathrm{conv}(E)$. For any subset Y of $\mathbb{R}^n$, the *affine hull* $\mathrm{aff}(Y)$ of Y is the set of all *affine combinations* in $\mathbb{R}^n$ of elements of Y. A set $\{y_1, \ldots, y_m\}$ of points in $\mathbb{R}^n$ is said to be *affinely independent* if none of its elements is an affine combination of the remaining elements. The *relative interior* $\mathrm{relint}(C)$ of a convex set $C \subseteq \mathbb{R}^n$ is the interior of C in the affine hull of C. For $0 \le d \le n$, a *d-simplex* T in $\mathbb{R}^n$ is the convex hull $\mathrm{conv}(v_0, \ldots, v_d)$ of $d+1$ affinely independent points in $\mathbb{R}^n$. The *vertices* $v_0, \ldots, v_d$ are uniquely determined by T. A *face* of T is the convex hull of a subset V of vertices of T. If the cardinality of V is d, then V is said to be a *facet* of T.

The *positive cone of $Y \subseteq \mathbb{R}^n$ at a point $x \in Y$* is the set

$$\mathrm{Cone}(Y, x) = \{y \in \mathbb{R}^n \mid x + \rho(y - x) \in Y \text{ for some } \rho > 0\}. \tag{6}$$

When T is a simplex, $\mathrm{Cone}(T,x)$ is closed. If F is a face of T and $x \in \mathrm{relint}(F)$ then for each $y \in F$ we have

$$\mathrm{Cone}(T,x) = \mathrm{aff}(F) + \mathrm{Cone}(T,y). \tag{7}$$

In particular, if $x \in \mathrm{relint}(T)$ then $\mathrm{Cone}(T,x) = \mathrm{aff}(T)$.

Lemma 3.1. *Suppose $T \subseteq \mathbb{R}^n$ is a simplex and F is a face of T.*

(a) *If S is an arbitrary simplex contained in T, and $F \cap \mathrm{relint}(S) \neq \emptyset$, then S is contained in F.*

(b) *A point z lies in $\mathrm{relint}(F)$ iff F is the smallest face of T containing z.*

Proof. (a) Let $F_1, \ldots, F_u$ be the facets of T, with their respective affine hulls $H_1, \ldots, H_u$. Each H_j is the boundary of the closed half-space $H_j^+ \subseteq T$ and of the other closed half-space H_j^-. Without loss of generality, $F_1, \ldots, F_t$ are the facets of T containing F. Then $\mathrm{aff}(F) = H_1 \cap \cdots \cap H_t$ and $F = (H_{t+1}^+ \cap \cdots \cap H_u^+) \cap \mathrm{aff}(F)$. By way of contradiction, suppose $x \in F \cap \mathrm{relint}(S)$ and $y \in S \setminus F$. For some $\epsilon > 0$ the segment $\mathrm{conv}(x + \epsilon(y - x), x - \epsilon(y - x))$ is contained in S. For some hyperplane $H \in \{H_1, \ldots, H_t\}$ the point y lies in the open half-space $\mathrm{int}(H^+) = \mathbb{R}^n \setminus H^-$, where "int" denotes topological interior. Now $x + \epsilon(y - x) \in \mathrm{int}(H^+)$ and $x - \epsilon(y - x) \in \mathrm{int}(H^-)$, whence $x - \epsilon(y - x) \notin T$, which contradicts $S \subseteq T$.

(b) This easily follows from (a). □

Proposition 3.2. *Let $x \in \mathbb{R}^n$ and $u_1, \ldots, u_m$ be linearly independent vectors in $\mathbb{R}^n$. Let $\lambda_1, \mu_1, \ldots, \lambda_m, \mu_m > 0$. Then the intersection of the two m-simplexes $\mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_m u_m)$ and $\mathrm{conv}(x, x + \mu_1 u_1, \ldots, x + \mu_1 u_1 + \cdots + \mu_m u_m)$ is an m-simplex of the form $\mathrm{conv}(x, x + \nu_1 u_1, \ldots, x + \nu_1 u_1 + \cdots + \nu_m u_m)$ for uniquely determined real numbers $\nu_1, \ldots, \nu_m > 0$.*

Proof. We argue by induction on $t = 1, \ldots, m$. The cases $t = 1, 2$ are trivial. Proceeding inductively, for any simplex $W = \mathrm{conv}(x, x + \theta_1 u_1, \ldots, x + \theta_1 u_1 + \cdots + \theta_t u_t)$, let $W' = \mathrm{conv}(x, x + \theta_1 u_1, \ldots, x + \theta_1 u_1 + \cdots + \theta_{t-1} u_{t-1})$ and $W'' = \mathrm{conv}(x, x + \theta_1 u_1, \ldots, x + \theta_1 u_1 + \cdots + \theta_{t-2} u_{t-2})$. By (7), for each $y \in W' \setminus W''$ the half-line from y in direction u_t intersects W in a segment $\mathrm{conv}(y, y + \gamma u_t)$ for some $\gamma > 0$ depending on y. Now let

$$U_t = \mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_t u_t),$$

$$V_t = \mathrm{conv}(x, x + \mu_1 u_1, \ldots, x + \mu_1 u_1 + \cdots + \mu_t u_t).$$

We then have

$$U_{t-1} = U_t' = \mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_{t-1} u_{t-1}),$$

$$V_{t-1} = V_t' = \mathrm{conv}(x, x + \mu_1 u_1, \ldots, x + \mu_1 u_1 + \cdots + \mu_{t-1} u_{t-1}),$$

and

$$U_{t-2} = U_t'' = \mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_{t-2} u_{t-2}),$$

$$V_{t-2} = V_t'' = \mathrm{conv}(x, x + \mu_1 u_1, \ldots, x + \mu_1 u_1 + \cdots + \mu_{t-2} u_{t-2}).$$

By induction hypothesis, for uniquely determined $\nu_1, \ldots, \nu_{t-1} > 0$ we can write

$$U_t' \cap V_t' = \mathrm{conv}(x, x + \nu_1 u_1, \ldots, x + \nu_1 u_1 + \cdots + \nu_{t-1} u_{t-1}).$$

The point $z = x + \nu_1 u_1 + \cdots + \nu_{t-1} u_{t-1}$ lies in $U_t' \setminus U_t''$. Let η_1 be the largest η such that $z + \eta u_t$ lies in U_t. Since $z \in V_t' \setminus V_t''$, let similarly η_2 be the largest η such that $z + \eta u_t$ lies in V_t. As already noted at the beginning of this proof, the real number $\nu_t = \min(\eta_1, \eta_2)$ is > 0. Evidently, ν_t is the largest η such that $z + \eta u_t$ lies in $U_t \cap V_t$. We conclude that $U_t \cap V_t = \mathrm{conv}(x, x + \nu_1 u_1, \ldots, x + \nu_1 u_1 + \cdots + \nu_t u_t)$. □

The following key result will find repeated use in the rest of this paper:

Theorem 3.3. *Let $(u_1, \ldots, u_k)$ be the Frenet k-frame of a sequence $\{x_i\}$ in $\mathbb{R}^n$ converging to x. Suppose a simplex $T \subseteq \mathbb{R}^n$ contains $\{x_i\}$. Then T contains the simplex $\mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k)$, for some $\lambda_1, \ldots, \lambda_k > 0$.*

Proof. We will prove the following stronger statement:

Claim. For each $l \in \{1, \ldots, k\}$ there exist $\lambda_1, \ldots, \lambda_l > 0$ such that:

(i) $\mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_l u_l) \subseteq T$ and

(ii) letting F_l be the smallest face of T containing the point $z_l = x + \lambda_1 u_1 + \cdots + \lambda_l u_l$ (which by Lemma 3.1(b) is equivalent to $z_l \in \mathrm{relint}(F_l)$), we have the inclusion $\mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_l u_l) \subseteq F_l$.

The proof is by induction on $l = 1, \ldots, k$.

Basis Step ($l = 1$)*:* Since each x_i is in T then $x + (x_i - x)/||x_i - x|| \in \mathrm{Cone}(T, x)$. Since $\mathrm{Cone}(T, x)$ is closed, then $x + u_1 \in \mathrm{Cone}(T, x)$. From (6) we obtain an $\epsilon > 0$ such that $x + \epsilon u_1 \in T$. Let $\lambda_1 = \epsilon/2$. Then $\mathrm{conv}(x, x + \lambda u_1) \subseteq \mathrm{conv}(x, x + \epsilon u_1) \subseteq T$, and (i) follows. Let F_1 be the smallest face of T containing the point $z_1 = x + \lambda_1 u_1$. Evidently, $z_1 \in \mathrm{relint}(\mathrm{conv}(x, x + \epsilon u_1))$. By Lemma 3.1(b), $z_i \in \mathrm{relint}(F_1)$. By Lemma 3.1(a), $F_1 \supseteq \mathrm{conv}(x, x + \epsilon u_1) \supseteq \mathrm{conv}(x, x + \lambda u_1)$. This proves (ii) and concludes the proof of the basis step.

Induction Step: For $1 \leq l < k$, induction yields $\lambda_1, \ldots, \lambda_l > 0$ such that, letting $C_l = \mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_l u_l)$ and $z_l = x + \lambda_1 u_1 + \cdots + \lambda_l u_l$, we have $C_l \subseteq T$. Further, letting F_l be the smallest face of T containing z_l, we have $C_l \subseteq F_l$, whence $\mathrm{aff}(C_l) = x + \mathbb{R}u_1 + \cdots + \mathbb{R}u_l \subseteq \mathrm{aff}(F_l)$. Since $z_l \in \mathrm{relint}(F_l)$ and $x_i - x \in \mathrm{Cone}(T, x)$, from (7) we obtain

$$z_l + \frac{x_i - x - \mathsf{proj}_{\mathbb{R}u_1 + \cdots + \mathbb{R}u_l}(x_i - x)}{||x_i - x - \mathsf{proj}_{\mathbb{R}u_1 + \cdots + \mathbb{R}u_l}(x_i - x)||} \in \mathrm{Cone}(T, z_l).$$

$\mathrm{Cone}(T, z_l)$ is closed, because $z_l + u_{l+1} \in \mathrm{Cone}(T, z_l)$. By (6), there exists $\epsilon > 0$ such that $z_l + \epsilon u_{l+1} \in T$, whence $\mathrm{conv}(z_l, z_l + \epsilon u_{l+1}) \subseteq T$. Setting now $\lambda_{l+1} = \epsilon/2$ and $z_{l+1} = z_l + \lambda_{l+1} u_{l+1}$, condition (i) in the claim above follows from the identity

$$\mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_{l+1} u_{l+1}) = \mathrm{conv}(C_l \cup \{z_{l+1}\}) \subseteq T.$$

Let F_{l+1} be the smallest face of T containing the point $z_{l+1} \in \mathrm{relint}(\mathrm{conv}(z_l, z_l + \epsilon u_{l+1}))$. By Lemma 3.1(b), $z_{l+1} \in \mathrm{relint}(F_{l+1})$. By Lemma 3.1(a),

$$F_{l+1} \supseteq \mathrm{conv}(z_l, z_l + \epsilon u_{l+1}) \supseteq \mathrm{conv}(z_l, z_l + \lambda_{l+1} u_{l+1}).$$

The minimality property of F_l yields $F_l \subseteq F_{l+1}$. By induction hypothesis, $C_l \subseteq F_{l+1}$. In conclusion, $\mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_{l+1} u_{l+1}) = \mathrm{conv}(C_l \cup \{z_{l+1}\}) \subseteq F_{l+1}$, as required to prove (ii) and to complete the proof. □

4. Tangents of X, principal ideals of $\mathcal{R}(X)$: the case $X \subseteq \mathbb{R}^2$

For $k = 1$ the following definition boils down to Definition 1.1 of Severi-Bouligand tangent vector. As in Definition 1.1, X is *an arbitrary* nonempty subset of $\mathbb{R}^n$.

Definition 4.1. Let $X \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $u = (u_1, \ldots, u_k)$ be a k-tuple of pairwise orthogonal unit vectors in $\mathbb{R}^n$. Then u is said to be a *tangent of* X *at* x if X contains a sequence $\{x_i\}$ converging to x, whose Frenet k-frame is u. We say that $\{x_i\}$ *determines* u. We say that u is *outgoing* if, in addition, there are $\lambda_1, \ldots, \lambda_k > 0$ such that the simplex $C = \mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k)$ and its facet $C' = \mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_{k-1} u_{k-1})$ have the same intersection with X.

The following elementary material on piecewise linear topology [12] is necessary to introduce the Riesz space $\mathcal{R}(X)$ of piecewise linear functions on X. In Theorem 5.1 below, the Frenet tangent frames of X will be related to the maximal and principal ideals of $\mathcal{R}(X)$.

A *polyhedron* P in $\mathbb{R}^n$ is a finite union of simplexes in $\mathbb{R}^n$. P need not be convex or connected. Given a polyhedron P, a *triangulation* of P is an (always finite) simplicial complex Δ such that $P = \bigcup \Delta$. Every polyhedron has a triangulation, [12, 2.1.5]. Given a rational polyhedron P and triangulations Δ and Σ of P, we say that Δ is a *subdivision* of Σ if every simplex of Δ is contained in a simplex of Σ. Suppose an n-cube $K \subseteq \mathbb{R}^n$ is contained in another n-cube $K' \subseteq \mathbb{R}^n$. Then every triangulation Δ of K has an *extension* Δ' to a triangulation of K', in the sense that $\Delta = \{T \in \Delta' \mid T \subseteq K\}$. A continuous function $f\colon K \to \mathbb{R}$ is Δ-*linear* if it is linear (in the affine sense) on each simplex of Δ. Via the extension Δ', f can be extended to a Δ'-linear function on K'. A function $g\colon K \to \mathbb{R}$ is *piecewise linear* if it is Δ-linear for some triangulation Δ of K. We denote by $\mathcal{R}(K)$ the Riesz space of all piecewise linear functions on K, with the pointwise operations of the Riesz space $\mathbb{R}$.

More generally, let X be a nonempty compact subset of $\mathbb{R}^n$. Let $K \subseteq \mathbb{R}^n$ be an (always closed) n-cube containing X. We momentarily denote by $\mathcal{R}(K)\restriction X$ the Riesz space of restrictions to X of the functions in $\mathcal{R}(K)$. If $L \subseteq \mathbb{R}^n$ is an n-cube containing K, then $\mathcal{R}(K)\restriction X = \mathcal{R}(L)\restriction X$. (For the nontrivial direction, the above mentioned extension property of triangulations yields $\mathcal{R}(L)\restriction K = \mathcal{R}(K)$.) Thus, if both n-cubes K and L contain X, letting $M \subseteq \mathbb{R}^n$ be an n-cube containing both K and L, we obtain $\mathcal{R}(K)\restriction X = \mathcal{R}(L)\restriction X = \mathcal{R}(M)\restriction X$, independently of the ambient cube $K \supseteq X$. Without fear of ambiguity we may then use the notation $\mathcal{R}(X)$ for the Riesz space of functions thus obtained. Each $f \in \mathcal{R}(X)$ is said to be a *piecewise linear function on X*. It follows that f is continuous.

Lemma 4.2. *There is a one-one correspondence $x \mapsto \mathfrak{m}_x$, $\mathfrak{m} \mapsto x_{\mathfrak{m}}$ between maximal ideals $\mathfrak{m}$ of $\mathcal{R}(X)$ and points x of X. Specifically, $\mathfrak{m}_x$ is the set of all functions in $\mathcal{R}(X)$ vanishing at x; conversely, $x_{\mathfrak{m}}$ is the only element in the intersection of the zerosets $Zh = h^{-1}(0)$ of all functions $h \in \mathfrak{m}$.*

Proof. The functions in $\mathcal{R}(X)$ separate points, and the constant function 1 is a strong unit in $\mathcal{R}(X)$. Now apply [6, 27.7]. □

The following elementary result deals with the special case $X \subseteq \mathbb{R}^2$. It is an adaptation to Riesz spaces of the MV-algebraic result [3, Theorem 3.1(ii)], and will have a key role in the proof of the much stronger Theorem 5.1.

Lemma 4.3. *Let $X \subseteq \mathbb{R}^2$ be a nonempty compact set. If the Riesz space $\mathcal{R}(X)$ has a principal ideal that is not an intersection of maximal ideals, then X has an outgoing Severi-Bouligand tangent at some point $x \in X$.*

Proof. For every element e of $\mathcal{R}(X)$ let $\langle e\rangle$ denote the principal ideal generated by e. Let $g \in \mathcal{R}(X)$ be such that the ideal $\mathfrak{p} = \langle g\rangle$ is not an intersection of maximal ideals of $\mathcal{R}(X)$. Lemma 4.2 yields an element $f \in \mathcal{R}(X)$ such that $f \notin \mathfrak{p}$ and $Zg \subseteq Zf$. Replacing, if necessary, f and g by their absolute values $|f|$ and $|g|$, we may assume $f \geq 0$ and $g \geq 0$. Let $K \subseteq \mathbb{R}^2$ be a fixed but otherwise arbitrary closed square containing X. By definition of $\mathcal{R}(X)$, there are elements $0 \leq \tilde{f} \in \mathcal{R}(K)$ and $0 \leq \tilde{g} \in \mathcal{R}(K)$ such that $\tilde{f}\restriction X = f$ and $\tilde{g}\restriction X = g$. Since $\tilde{f}\restriction X$ does not belong to $\mathfrak{p}$ then for each $m > 0$ there is a point $x_m \in X$ such that

$$\tilde{f}(x_m) > m \cdot \tilde{g}(x_m). \tag{8}$$

Since X is compact, for some $x \in X$ there is a subsequence $\{x_{m_1}, x_{m_2}, \ldots\}$ of $\{x_1, x_2, \ldots\}$ such that

$$x_i \neq x_j \text{ for all } i \neq j, \quad \text{and} \quad \lim_{i\to\infty} x_{m_i} = x. \tag{9}$$

For each $i = 1, 2, \ldots$, let the unit vector u_i be defined by

$$u_i = (x_{m_i} - x)/||x_{m_i} - x||.$$

Since the unit circumference $S^1 = \{z \in \mathbb{R}^2 \mid ||z|| = 1\}$ is compact, it is no loss of generality to assume $\lim_{i\to\infty} u_i = u$, for some $u \in S^1$. Therefore, u is a tangent of X at x. There remains to be shown that u is outgoing. To this purpose we make the following

Claim. There is a real number $\lambda > 0$ such that:

(a) $\tilde{f}$ is (affine) linear on the line segment $\mathrm{conv}(x, x + \lambda u)$;
(b) $\tilde{g}$ identically vanishes on $\mathrm{conv}(x, x + \lambda u)$;
(c) $\tilde{f}(x + \lambda u) \neq 0$.

As a matter of fact, since each of $x_{m_1}, x_{m_2}, \ldots$ lies in K, by (9) there exists $\delta > 0$ such that $\mathrm{conv}(x, x + \delta u) \subseteq K$. An elementary result in polyhedral topology ([12, 2.2.4]) yields a triangulation Δ of K such that both functions $\tilde{f}$ and $\tilde{g}$ are Δ-linear and $\mathrm{conv}(x, x + \delta u) = \bigcup\{T \in$

$\Delta \mid T \subseteq \mathrm{conv}(x, x+\delta u)\}$. Therefore, there exists $\lambda > 0$ such that $\mathrm{conv}(x, x+\lambda u) \in \Delta$. We have proved that $\tilde{f}$ is linear in $\mathrm{conv}(x, x+\lambda u)$, and (a) is settled.

To settle (b), since both functions $\tilde{g}$ and $\tilde{f}$ are continuous, we can write

$$0 \geq \tilde{g}(x) \;=\; \lim_{i\to\infty} \tilde{g}(x_i) \;\leq\; \lim_{i\to\infty} \frac{\tilde{f}(x_i)}{m_i} \;=\; 0,$$

whence $\tilde{g}(x) = g(x) = 0$. From $X \cap Z\tilde{g} \subseteq X \cap Z\tilde{f}$ we get $\tilde{f}(x) = f(x) = 0$. Since Δ is finite set, there exists a 2-simplex $S \in \Delta$ containing infinitely many elements $x_{n_1}, x_{n_2}, \ldots$ of the set $\{x_{m_1}, x_{m_2}, \ldots\}$. By (9), $x \in S$. Further, from $\lim_{i\to\infty} u_{n_i} = u$ and $\mathrm{conv}(x, x+\lambda u) \in \Delta$ it follows that $\mathrm{conv}(x, x+\lambda u) \subseteq S$. Therefore,

$$S = \mathrm{conv}(x, x+\lambda u, v) \text{ for some } v \in S. \tag{10}$$

For some 2×1-matrix A and vector $b \in \mathbb{R}^2$ we can write $\tilde{g}(z) = Az + b$ for each $z \in S$. Since $\lim_{i\to\infty} u_{m_i} = u$ and $\tilde{g}(x) = 0$, we have the identities

$$\begin{aligned}\tilde{g}(x+\lambda u) = \lambda A u + \tilde{g}(x) &= \lim_{i\to\infty} \frac{\lambda(Ax_{n_i} - Ax)}{||x_{n_i} - x||} = \lim_{i\to\infty} \frac{\lambda(\tilde{g}(x_{n_i}) - \tilde{g}(x))}{||x_{n_i} - x||}\\ &= \lim_{i\to\infty} \frac{\lambda \tilde{g}(x_{n_i})}{||x_{n_i} - x||} = \lim_{i\to\infty} \frac{\lambda g(x_{n_i})}{||x_{n_i} - x||}.\end{aligned}$$

Similarly,

$$\tilde{f}(x+\lambda u) = \lim_{i\to\infty} \frac{\lambda f(x_{n_i})}{||x_{n_i} - x||},$$

whence

$$0 \leq \tilde{g}(x+\lambda u) = \lim_{i\to\infty} \frac{\lambda g(x_{n_i})}{||x_{n_i} - x||} \leq \lim_{i\to\infty} \frac{\lambda}{n_i} \frac{f(x_{n_i})}{||x_{n_i} - x||} = \tilde{f}(x+\lambda u) \lim_{i\to\infty} \frac{1}{n_i} = 0.$$

Since $\tilde{g}$ is linear on $\mathrm{conv}(x, x+\lambda u)$ and $\tilde{g}(x+\lambda u) = 0 = \tilde{g}(x)$, then (b) follows.

To prove (c), by (8) we get $\tilde{f}(x_{n_i}) \neq 0$ for all i, whence $\tilde{g}(x_{n_i}) \neq 0$, because $Zg \subseteq Zf$. Then our assumptions about S, together with (10), show that $\tilde{g}(v) \neq 0$. Let the integer m^* satisfy the inequality $m^* \cdot \tilde{g}(v) \geq \tilde{f}(v)$. If (absurdum hypothesis) $\tilde{f}(x+\lambda u) = 0$ then $m^* \cdot \tilde{g}(z) \geq \tilde{f}(z)$ for each $z \in S$. In view of (8), this contradicts the existence of infinitely many elements x_{n_i} in S. Having thus proved (c), our claim is settled.

In conclusion, from (a) and (c) it follows that $\mathrm{conv}(x, x+\lambda u) \cap Z\tilde{f} = \{x\}$. Then from (b) we get

$$X \cap \mathrm{conv}(x, x+\lambda u) = X \cap Z\tilde{g} \cap \mathrm{conv}(x, x+\lambda u) \subseteq X \cap Z\tilde{f} \cap \mathrm{conv}(x, x+\lambda u) = \{x\},$$

thus proving that u is an outgoing tangent of X at x. □

5. Tangents and strong semisimplicity

Recall that a Riesz space R is said to be *strongly semisimple* if for every principal ideal $\langle g\rangle$ of R the quotient $R/\langle g\rangle$ is *archimedean* (i.e., the intersection of the maximal ideals of $R/\langle g\rangle$ is $\{0\}$). Equivalently, $\langle g\rangle$ is an intersection of maximal ideals of R. (This follows from the canonical one-to-one correspondence between ideals of R containing $\langle g\rangle$, and ideals of $R/\langle g\rangle$.) Since $\{0\}$ is a principal ideal of R, if R is strongly semisimple then it is archimedean.

The following result is the promised strengthening of Lemma 4.3:

Theorem 5.1. *For any nonempty compact set $X \subseteq \mathbb{R}^n$ the following conditions are equivalent:*

(i) *X has an outgoing tangent at some point $x \in X$.*

(ii) *The Riesz space $\mathcal{R}(X)$ is not strongly semisimple, i.e., there exists a principal ideal of $\mathcal{R}(X)$ that is not an intersection of maximal ideals.*

Proof. Without loss of generality, $X \subseteq [0,1]^n$. (This trivially follows because any n-cube in $\mathbb{R}^n$ is PL-homeomorphic to any other n-cube).

(i)⇒(ii) By Definition 4.1, for some $x \in \mathbb{R}^n$ and k-tuple $u = (u_1, \dots, u_k)$ of pairwise orthogonal unit vectors in $\mathbb{R}^n$, there is a sequence $\{x_i\}$ of points in $\mathbb{R}^n$ converging to x, such that u is the Frenet k-frame of $\{x_i\}$. Further, there are reals $\lambda_1, \dots, \lambda_k > 0$ such that the simplex $C = \mathrm{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k)$ and its facet $C' = \mathrm{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_{k-1} u_{k-1})$ satisfy $C \cap X = C' \cap X$.

Let f_1 and f_2 be piecewise linear functions defined on $[0,1]^n$, taking their values in $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$ and satisfying the conditions

$$Zf_1 = f_1^{-1}(0) = C, \quad Zf_2 = C', \text{ and } \quad f_2 \text{ is (affine) linear over } C. \tag{11}$$

The existence of f_1 and f_2 follows from [12, 2.2.4]. Both restrictions $f_2 \restriction X$ and $f_1 \restriction X$ are elements of $\mathcal{R}(X)$. By construction,

$$Zf_1 \cap X = Zf_2 \cap X. \tag{12}$$

We *claim* that the principal ideal $\mathfrak{p} = \langle f_1 \restriction X \rangle$ of $\mathcal{R}(X)$ generated by $f_1 \restriction X$ does not coincide with the intersection of all maximal ideals of $\mathcal{R}(X)$ containing $\mathfrak{p}$.

By (12) together with Lemma 4.2, $f_2 \restriction X$ belongs to all maximal ideals of $\mathcal{R}(X)$ containing $\mathfrak{p}$. So our claim will be settled once we prove

$$f_2 \restriction X \notin \mathfrak{p}. \tag{13}$$

To this purpose, arguing by way of contradiction, suppose $f_2 \restriction X \leq m f_1 \restriction X$ for some $m = 1, 2, \dots$. Since f_1 and f_2 are (continuous) piecewise linear, the set $L = \{x \in [0,1]^n \mid f_2(x) \leq m f_1(x)\}$ is a union of simplexes $T_1 \cup \dots \cup T_r$. Necessarily for some $j = 1, \dots, r$ the simplex T_j contains infinitely many points of the sequence $\{x_i\}$. This subsequence $\{x_t\}$ still converges to $x \in T_j$, and u is its Frenet k-frame. Theorem 3.3 yields $\mu_1, \dots, \mu_k > 0$ such that T_j contains the simplex $M = \mathrm{conv}(x, x + \mu_1 u_1, \dots, x + \mu_1 u_1 + \dots + \mu_k u_k)$. Now Proposition 3.2 yields uniquely determined $\nu_1, \dots, \nu_k > 0$ such that

$$C \cap M = \mathrm{conv}(x, x + \nu_1 u_1, \dots, x + \nu_1 u_1 + \dots + \nu_k u_k).$$

By (11), f_1 identically vanishes on $C \cap M$. Further, from $L \supseteq T_j \supseteq M \supseteq C \cap M$ and $f_2 \leq m f_1$ on L, it follows that $f_2 = 0$ on $C \cap M$. The two simplexes $C \cap M$ and C have the same dimension k, and f_2 is (affine) linear on $C \supseteq C \cap M$. Therefore, $f_2 = 0$ on C, which contradicts $Zf_2 = C'$. We have thus proved (13), settled our claim, and completed the proof of (i)⇒(ii).

(ii)⇒(i) By hypothesis, there is a function $f_1 \in \mathcal{R}([0,1]^n)$ such that the principal ideal $\langle f_1 \restriction X \rangle$ of $\mathcal{R}(X)$ generated by the restriction $f_1 \restriction X$ is not an intersection of maximal ideals of $\mathcal{R}(X)$. Thus there is $f_2 \in \mathcal{R}([0,1]^n)$ whose restriction $f_2 \restriction X$ does not belong to the principal ideal $\langle f_1 \restriction X \rangle$ generated by $f_1 \restriction X$, but belongs to all maximal ideals of $\mathcal{R}(X)$ containing $\langle f_1 \restriction X \rangle$. By Lemma 4.2, $Zf_2 \restriction X = Zf_1 \restriction X$, i.e., $X \cap Zf_2 = X \cap Zf_1$.

Let the map $g \colon X \to \mathbb{R}^2$ be defined by

$$g(x) = (f_1(x), f_2(x)) \text{ for all } x \in X. \tag{14}$$

Let $\iota \colon \mathcal{R}(g(X)) \to \mathcal{R}(X)$ be defined by $\iota(h) = h \circ g$ for all $h \in \mathcal{R}(g(X))$, where $\circ$ denotes composition. It is easy to see that ι is a Riesz space homomorphism of $\mathcal{R}(g(X))$ into $\mathcal{R}(X)$. Letting $\pi_1, \pi_2 \colon \mathbb{R}^2 \to \mathbb{R}$ be the canonical projections (=coordinate functions), we have the identities $f_1 \restriction X = \iota(\pi_1 \restriction g(X))$ and $f_2 \restriction X = \iota(\pi_2 \restriction g(X))$. Whenever $h \in \mathcal{R}(g(X))$, $\iota(h) = 0$ and $z \in g(X)$, there exists $x \in X$ such that $g(x) = z$. Then $h(z) = h(g(x)) = (\iota(h))(x) = 0$ and ι is one-to-one. Actually, ι is an isomorphism between $\mathcal{R}(g(X))$ and the Riesz subspace of $\mathcal{R}(X)$ generated by $\{f_1 \restriction X, f_2 \restriction X\}$. It follows that the principal ideal $\mathfrak{p}$ of $\mathcal{R}(g(X))$ generated by $\pi_1 \restriction g(X)$ is not an intersection of maximal ideals of $\mathcal{R}(g(X))$: specifically, $\pi_2 \restriction g(X)$ belongs to all maximal ideals containing $\mathfrak{p}$, but does not belong to $\mathfrak{p}$. By Lemma 4.3,

$$g(X) \text{ has a Severi-Bouligand outgoing tangent.} \tag{15}$$

There remains to be proved that X has an outgoing tangent. To help the reader, the long proof is subdivided into two parts.

$\boxed{\text{Part 1: Construction of a tangent } u \text{ of } X.}$

By (15) and Definition 4.1 with $k = 1$ (which is the same as Definition 1.1), for some point $y^* \in \mathbb{R}^2$, unit vector $v^* \in \mathbb{R}^2$, sequence $\{y_i\} \subseteq \mathbb{R}^2$ converging to y^*, and $\mu > 0$, we can write

$$\lim_{i\to\infty}(y_i - y^*)/||y_i - y^*|| = v^* \quad \text{and} \quad \mathrm{conv}(y^*, y^* + \mu v^*) \cap g(X) = \{y^*\}. \tag{16}$$

By (14), g is the restriction to X of the function $f = (f_1, f_2)\colon [0,1]^n \to \mathbb{R}^2$. Since (each component of) f is piecewise linear, then f is continuous, and both sets $f^{-1}(y^*)$ and $f^{-1}(\mathrm{conv}(y^*, y^* + \mu v^*))$ are polyhedra in $[0,1]^n$. An elementary result in polyhedral topology ([12, 2.2.4]) yields a triangulation Δ of $[0,1]^n$ having the following properties:

- f is (affine) linear over each simplex of Δ,
- $f^{-1}(y^*) = \bigcup\{R \in \Delta \mid R \subseteq f^{-1}(y^*)\}$, and
- $f^{-1}(\mathrm{conv}(y^*, y^* + \mu v^*)) = \bigcup\{U \in \Delta \mid U \subseteq f^{-1}(\mathrm{conv}(y^*, y^* + \mu v^*))\}$.

For some n-simplex $T \in \Delta$, the set $\{i \mid f^{-1}(y_i) \cap T \cap X\} = \{i \mid g^{-1}(y_i) \cap T\}$ is infinite. Let $z_0, z_1, \ldots$ be a converging sequence of elements of T such that $f(z_0), f(z_1), \ldots$ is a subsequence of $y_0, y_1, \ldots$. Without loss of generality this subsequence coincides with the sequence $\{y_i\}$, and we can write

$$g(z_i) = y_i. \tag{17}$$

Letting $z^* = \lim_{i\to\infty} z_i$ we have

$$z^* \in X \cap T \quad \text{and} \quad y^* = f(z^*) = g(z^*). \tag{18}$$

The linearity of f on T yields a $2\times n$ matrix A, together with a vector $b \in \mathbb{R}^2$ such that for each $t \in T$, $f(t) = At + b$.

Claim. For some $k \in \{1, \ldots, n\}$ there is a k-tuple of pairwise orthogonal unit vectors $u_i \in \mathbb{R}^n$, $(1 \le i \le k)$ such that:

- $Au_j = 0$ for each $1 \le j < k$,
- $Au_k \neq 0$,
- $u = (u_1, \ldots, u_k)$ is a tangent of X at z^*, determined by a suitable subsequence of $z_0, z_1, \ldots$, in the sense of Definition 4.1.

The vectors $u_1, \ldots, u_k$ are constructed by the following inductive procedure:

Basis Step: From $Az_i + b = y_i \neq y^* = Az^* + b$ it follows that $z_i \neq z^*$ for each i, and hence every vector $z_i^1 = (z_i - z^*)/||z_i - z^*||$ is well defined. Since the $(n-1)$-dimensional unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ is compact, it is no loss of generality to assume that the sequence $z_0^1, z_1^1, \ldots$ converges to some unit vector u_1. It follows that u_1 is a tangent of X at z^*. If $Au_1 \neq 0$, upon setting $u = u_1$ the claim is proved. If $Au_1 = 0$ we proceed inductively.

Induction Step: Having constructed a tangent $u(l) = (u_1, \ldots, u_l)$ of X at z^* with $Au_i = 0$ for each $i \in \{1, \ldots, l\}$, we first observe that $l < n$. (For otherwise, the u_j would constitute an orthonormal basis of $\mathbb{R}^n$, whence A is the zero matrix, and $Ax + b = b$ for each $x \in \mathbb{R}^n$, which contradicts $Az_i + b \neq Az^* + b$.) Let $\rho_1, \ldots, \rho_l$ be arbitrary real numbers. From

$$A(z^* + \rho_1 u_1 + \cdots + \rho_l u_l) + b = A(z^*) + b = g(z^*) \neq g(z_i) = A(z_i) + b, \tag{19}$$

it follows that no z_i lies in the affine space $z^* + \mathbb{R}u_1 + \cdots + \mathbb{R}u_l$, i.e., $z_i - z^* \notin \mathbb{R}u_1 + \cdots + \mathbb{R}u_l$. For each i, the unit vector

$$z_i^{l+1} = \frac{z_i - z^* - \mathsf{proj}_{\mathbb{R}u_1+\cdots+\mathbb{R}u_l}(z_i - z^*)}{||z_i - z^* - \mathsf{proj}_{\mathbb{R}u_1+\cdots+\mathbb{R}u_l}(z_i - z^*)||}$$

is well defined. Without loss of generality, we can write $\lim_{i\to\infty} z_i^{l+1} = u_{l+1}$ for some unit vector $u_{l+1} \in \mathbb{R}^n$. By construction, u_{l+1} is orthogonal to each of $u_1, \ldots, u_l$, and the $(l+1)$-tuple

$u(l+1) = (u_1, \dots, u_l, u_{l+1})$ is a tangent of X at z^*. In case $Au_{l+1} \neq 0$, upon setting $k = l+1$ and $u = u(l+1)$ we are done. In case $Au_{l+1} = 0$, we proceed inductively, with $(u_1, \dots, u_l, u_{l+1})$ in place of $(u_1, \dots, u_l)$. Our claim is settled, and so is the proof of Part 1.

Part 2: u is an outgoing tangent of X.

With the notation of Part 1, for some $\lambda_1, \dots, \lambda_k > 0$ we prove the inclusion

$$\mathrm{conv}(z^*, z^* + \lambda_1 u_1, \dots, z^* + \lambda_1 u_1 + \cdots + \lambda_k u_k) \subseteq T \cap f^{-1}(\mathrm{conv}(y^*, y^* + \mu v^*)). \tag{20}$$

As a matter of fact, by construction, $u = (u_1, \dots, u_k)$ is a tangent of $X \cap T$ at z^*. Theorem 3.3 yields real numbers $\epsilon_1, \dots, \epsilon_k > 0$ such that

$$\mathrm{conv}(z^*, z^* + \epsilon_1 u_1, \dots, z^* + \epsilon_1 u_1 + \cdots + \epsilon_k u_k) \subseteq T. \tag{21}$$

Since $Au_j = 0$ for each $j = 1, \dots, k-1$, from (18)-(19) we obtain the identities

$$y^* = g(z^*) = g(x) \text{ for all } x \in \mathrm{conv}(z^*, z^* + \epsilon_1 u_1, \dots, z^* + \epsilon_1 u_1 + \cdots + \epsilon_{k-1} u_{k-1}). \tag{22}$$

Recalling (17) we can write

$$\begin{aligned} 0 \neq Au_k &= \lim_{i\to\infty} Az_i^k = \lim_{i\to\infty} A\left(\frac{z_i - z^* - \mathsf{proj}_{\mathbb{R}u_1+\cdots+\mathbb{R}u_{l-1}}(z_i - z^*)}{||z_i - z^* - \mathsf{proj}_{\mathbb{R}u_1+\cdots+\mathbb{R}u_{l-1}}(z_i - z^*)||}\right) \\ &= \lim_{i\to\infty} \frac{A(z_i) - A(z^*)}{||z_i - z^* - \mathsf{proj}_{\mathbb{R}u_1+\cdots+\mathbb{R}u_{l-1}}(z_i - z^*)||} \\ &= \lim_{i\to\infty} \frac{y_i - y^*}{||z_i - z^* - \mathsf{proj}_{\mathbb{R}u_1+\cdots+\mathbb{R}u_{l-1}}(z_i - z^*)||} \cdot \frac{||y_i - y^*||}{||y_i - y^*||} \\ &= \lim_{i\to\infty} \frac{y_i - y^*}{||y_i - y^*||} \cdot \frac{||y_i - y^*||}{||z_i - z^* - \mathsf{proj}_{\mathbb{R}u_1+\cdots+\mathbb{R}u_{l-1}}(z_i - z^*)||}. \end{aligned}$$

Since $0 \neq v^* = \lim_{i\to\infty}(y_i - y^*)/||y_i - y^*||$, for some $\tau > 0$ we obtain

$$\tau = \lim_{i\to\infty} \frac{||y_i - y^*||}{||z_i - z^* - \mathsf{proj}_{\mathbb{R}u_1+\cdots+\mathbb{R}u_{l-1}}(z_i - z^*)||} \quad \text{and} \quad Au_k = \tau v^*.$$

Now the desired λ's in (20) are given by setting $\lambda_j = \epsilon_j$ for $1 \le j < k$, and $\lambda_k = \min\{\epsilon_k, \mu/\tau\}$. Indeed, letting $C = \mathrm{conv}(z^*, z^* + \lambda_1 u_1, \dots, z^* + \lambda_1 u_1 + \cdots + \lambda_k u_k)$, from (21) we obtain

$$C \subseteq \mathrm{conv}(z^*, z^* + \epsilon_1 u_1, \dots, z^* + \epsilon_1 u_1 + \cdots + \epsilon_k u_k) \subseteq T. \tag{23}$$

Further, for every $x \in C$ there exists $0 \le \omega \le \lambda_k$ such that

$$Ax + b = Az^* + \omega Au_k + b = Az^* + b + \omega\tau v^* = y^* + \omega\tau v^*, \tag{24}$$

whence $Ax + b \in \mathrm{conv}(y^*, y^* + \mu v^*)$, because $\omega \le \mu/\tau$. The proof of (20) is complete.

To complete the proof that $(u_1, \dots, u_k)$ is outgoing, letting $C' = \mathrm{conv}(z^*, z^* + \lambda_1 u_1, \dots, z^* + \lambda_1 u_1 + \cdots + \lambda_{k-1} u_{k-1})$, we must show $C' \cap X = C \cap X$. By way of contradiction, suppose $x \in (X \cap C) \setminus (X \cap C')$. Then for suitable $\xi_1, \dots, \xi_{k-1} \ge 0$ and $\xi_k > 0$, we can write $x = z^* + \xi_1 u_1 + \cdots + \xi_k u_k$. By (23), $x \in X \cap T$. Since $\xi_k > 0$, by (24) we have $g(x) = f(x) = Ax + b = y^* + \xi_k \tau v^* \neq y^*$. This contradicts the identity $g(x) \in g(X) \cap \mathrm{conv}(y^*, y^* + \mu v^*) = \{y^*\}$, which follows from (16) and (22).

Having thus proved that the tangent u is outgoing, we have also completed the proof of Part 2, as well as the proof of the theorem. □

6. Examples and Further Results

Proposition 6.1. *Let $I = \mathrm{conv}(a,b) \subseteq \mathbb{R}$ be an interval, and $\phi\colon I \to \mathbb{R}^n$ a C^2 function. Then the Riesz space $\mathcal{R}(\phi(I))$ is strongly semisimple iff ϕ is (affine) linear.*

Proof. The proof directly follows from Theorems 5.1 and 2.2. □

Proposition 6.2. *For every polyhedron $P \subseteq \mathbb{R}^n$ the Riesz space $\mathcal{R}(P)$ is strongly semisimple, and P has no outgoing tangent.*

Proof. For some finite set $\{S_1, \ldots, S_m\}$ of simplexes in $\mathbb{R}^n$ we can write $P = S_1 \cup \cdots \cup S_m$. If u is a tangent of P at some point $x \in P$ then u is also a tangent of S_i at x for some $i = 1, \ldots, m$. By Theorem 3.3, u is not an outgoing tangent of S_i. Thus u is not an outgoing tangent of P. Now apply Theorem 5.1. □

The following is an example of a strongly semisimple Riesz space $\mathcal{R}(X)$, where X is not a polyhedron:

Example 6.3. Let the set $X \subseteq \mathbb{R}^2$ be defined by

$$X = \{(0,0)\} \cup \{(1/n, 0) \mid n = 1, 2, \ldots\} \cup \{(1/n, 1/n^2) \mid n = 1, 2, \ldots\}.$$

The origin $(0,0)$ is the only accumulation point of X. The only tangents of X are given by the vector $(1,0)$ and the pair of vectors $((1,0),(0,1))$. Therefore, X has no outgoing tangents. By Theorem 5.1, the Riesz space $\mathcal{R}(X)$ is strongly semisimple.

However, when the compact set $X \subseteq \mathbb{R}^n$ is convex we have:

Theorem 6.4. *Let $X \subseteq \mathbb{R}^n$ be a nonempty compact convex set. Then the following conditions are equivalent:*

(I) *The Riesz space $\mathcal{R}(X)$ is strongly semisimple.*
(II) *$X = \mathrm{conv}(x_1, \ldots, x_m)$ for some $x_1, \ldots, x_m \in \mathbb{R}^n$, i.e., X is a polyhedron.*
(III) *X has no outgoing tangent.*

Proof. (III)⇔(I) This is a particular case of Theorem 5.1. (II)⇒(I) By Proposition 6.2. (I)⇒(II) Arguing by way of contradiction, assume $\mathcal{R}(P)$ to be strongly semisimple, but $X \neq \mathrm{conv}(x_1, \ldots, x_m)$ for any finite set $\{x_1, \ldots, x_m\} \subseteq \mathbb{R}^n$. Letting $\mathrm{ext}(X)$ denote the set of extreme point of X, Minkowski theorem yields the identity $X = \mathrm{conv}(\mathrm{ext}(X))$. Since X is compact, there exists a point $x \in X$ together with a sequence $x_1, x_2, \ldots$ of extreme points of X such that $\lim_{i\to\infty} x_i = x$ and $x_i \neq x_j$ for every $i \neq j$.

Claim 1. There exists a subsequence $x_{m_1}, x_{m_2}, \ldots$ of the sequence $x_1, x_2, \ldots$, together with a k-tuple $(u_1, \ldots, u_k)$ of pairwise orthogonal unit vectors in $\mathbb{R}^n$ (for some $k \in \{1, \ldots, n\}$), having the following properties:

(a) $x_{m_1}, x_{m_2}, \ldots$ determines the tangent $(u_1, \ldots, u_k)$ of X at x, in the sense of Definition 4.1.
(b) $\mathrm{aff}(x_{m_1}, x_{m_2}, \ldots) = x + \mathbb{R}u_1 + \cdots + \mathbb{R}u_k$.

The vectors $u_1, u_2, \ldots, u_k$ are constructed by the following inductive procedure:

Basis: Since $x_i \neq x_j$ for each $i \neq j$, then each unit vector $(x_i - x)/||x_i - x||$ is well defined. There is a subsequence $x_{m_1^1}, x_{m_2^1}, \ldots$ of $x_1, x_2, \ldots$ and a unit vector $u_1 \in \mathbb{R}^n$ such that $\lim_{i\to\infty}(x_{m_i^1} - x)/||x_{m_i^1} - x|| = u_1$. Then u_1 is a tangent of X at x determined by $x_{m_1^1}, x_{m_2^1}, \ldots$.

Induction Step: Let $l \geq 1$ and assume the subsequence $x_{m_1^l}, x_{m_2^l}, \ldots$ of $x_1, x_2, \ldots$ determines the tangent $(u_1, \ldots, u_l)$ of X at x. If there exists an integer r such that $\mathrm{aff}(x_{m_r^l}, x_{m_{r+1}^l}, \ldots) = x + \mathbb{R}u_1 + \cdots + \mathbb{R}u_l$, then upon setting $k = l$, we are done. If no such r exists, infinitely many vectors in $x_{m_1^l}, x_{m_2^l}, \ldots$ do not belong to the affine space $x + \mathbb{R}u_1 + \cdots + \mathbb{R}u_l$. Therefore, for some subsequence $x_{m_1^{l+1}}, x_{m_2^{l+1}}, \ldots$ and unit vector $u_{l+1} \in \mathbb{R}^n$ we can write

$$u_{l+1} = \lim_{i\to\infty} \frac{x_{m_i^{l+1}} - x - \mathsf{proj}_{\mathbb{R}u_1+\cdots+\mathbb{R}u_l}(x_{m_i^{l+1}} - x)}{||x_{m_i^{l+1}} - x - \mathsf{proj}_{\mathbb{R}u_1+\cdots+\mathbb{R}u_l}(x_{m_i^{l+1}} - x)||}. \tag{25}$$

We then proceed with $(u_1, \ldots, u_{l+1})$ in place of $(u_1, \ldots, u_l)$. Since the affine space $\mathrm{aff}(x_{m_1}, x_{m_2}, \ldots)$ is contained in $\mathbb{R}^n$, this procedure must terminate for some $1 \le k \le n$. Claim 1 is settled.

Let us now fix a subsequence $x_{m_1}, x_{m_2}, \ldots$ of $x_1, x_2, \ldots,$ together with a k-tuple $(u_1, \ldots, u_k)$ of pairwise orthogonal unit vectors satisfying conditions (a) and (b) in Claim 1.

Claim 2. There are $\lambda_1, \ldots, \lambda_k > 0$ such that the k-simplex $C_k = \mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_k u_k)$ is contained in X.

We have already observed that $x \in X$. By Theorem 5.1, the tangent u_1 of X at x is not outgoing. Hence $\mathrm{conv}(x, x + u_1) \cap X \neq \{x\}$. Let $y \in (\mathrm{conv}(x, x + u_1) \cap X) \setminus \{x\}$. Thus $y = x + \lambda_1 u_1$ for some $0 < \lambda_1 \le 1$. Since X is convex, $\mathrm{conv}(x, x + \lambda_1 u_1) \subseteq X$.

Proceeding inductively, let us assume that $\lambda_1, \ldots, \lambda_l > 0$ are such that the l-simplex $C_l = \mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_l u_l)$ is contained in X, for some $l \in \{1, \ldots, k\}$. If $l = k$ we are done. If $l < k$ let $C'_{l+1} = \mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_l u_l + u_{l+1})$. By construction, $(u_1, \ldots, u_{l+1})$ is a tangent of X at x. Since by hypothesis $\mathcal{R}(X)$ is strongly semisimple, by Theorem 5.1 $(u_1, \ldots, u_{l+1})$ is not outgoing, whence there is $y \in (C'_{l+1} \cap X) \setminus C_l$. As a consequence, there are $\lambda'_1, \ldots, \lambda'_l > 0$ and $\lambda_{l+1} > 0$ such that $y = x + \lambda'_1 u_1 + \cdots + \lambda'_l u_l + \lambda_{l+1} u_{l+1}$ and $\lambda'_i \le \lambda_i$ for each $i \in \{1, \ldots, l\}$. Since X is convex, the set $\mathrm{conv}(x,\ x + \lambda'_1 u_1, \ldots, x + \lambda'_1 u_1 + \cdots + \lambda'_l u_l,\ y)$ is contained in X. Setting now (without loss of generality) $\lambda_i = \lambda'_i$, we obtain the inclusion $C_{l+1} = \mathrm{conv}(x, x + \lambda_1 u_1, \ldots, x + \lambda_1 u_1 + \cdots + \lambda_{l+1} u_{l+1}) \subseteq X$, thus completing the inductive step. This procedure terminates after k steps. Claim 2 is settled.

Since the k-simplex C_k is contained in the affine space $\mathrm{aff}(x_{m_1}, x_{m_2}, \ldots)$, and $(u_1, \ldots, u_k)$ is the Frenet k-frame of the sequence $x_{m_1}, x_{m_2}, \ldots$, the exists an integer $r^* > 0$ such that $x_{m_j} \in C_k$ for each $j = r^*, r^* + 1, \ldots$. By definition, $x_{m_1}, x_{m_2}, \ldots \in \mathrm{ext}(X)$. By Claim 2, $C_k \subseteq X$. Thus $x_{m_{r^*}}, x_{m_{r^*+1}}, \ldots \in \mathrm{ext}(C_k)$. Since $x_i \neq x_j$ for every $i \neq j$, then the set $\mathrm{ext}(C_k)$ must be infinite, a contradiction. The proof is complete. □

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